A TREE CAN MAKE A DIFFERENCE[1]

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We show that it is not possible to extend the ranking of one-stage lotteries based on their weak-expectation to a reflexive and transitive (but possibly incomplete) relation on the collection of one- and two-stage lotteries that satisfies two basic axioms: the minimal value axiom and the reduction axiom. We propose an extension that satisfies only the first axiom. This ranking takes payoffs, their probabilities, and the tree structure into account.

I. INTRODUCTION

By appealing to the weak law of large numbers, Kenny Easwaran has extended the expectation-based ranking of lotteries (obeying the strong law of large numbers) to the class of weakly integrable lotteries.[2] Not surprisingly, such an extension entails a loss: not all properties satisfied by the standard expectation-based ranking carry over to the larger domain.

In this note, we employ the framework of one-stage and two-stage lotteries and focus on a reduction axiom and a minimal value axiom. The *reduction axiom* is outcome oriented and judges a two-stage lottery solely on the basis of the final outcomes and their compound probabilities.[3] The *minimal value axiom* states that a two-stage lottery cannot have a value smaller than the smallest of the various expected values of the second-stage lotteries. The

[1] We give special thanks to the referee for extremely helpful comments.
[2] Kenny Easwaran, "Strong and Weak Expectations," *Mind*, CXVII, (July 2008): 633-41.
[3] Uzi Segal, "Two-stage Lotteries without the Reduction Axiom," *Econometrica*, LVIII, (March 1990): 349-377.

ranking of options on the basis of their *standard* expected values satisfies both axioms. In contrast, the ranking based on weak expected values (defined below) does not.

Those who endorse weak-expectation decision theory must therefore reject or weaken at least one of the two remaining axioms. Rejecting the minimal value axiom entails the paradox of a two-stage lottery with a value smaller than the smallest expected value of the second-stage lotteries. Rejecting the reduction axiom entails the fact that payoffs and their probabilities are not sufficient for the assessment of lotteries. We support the latter view, and we will argue that the *tree-structure* behind lotteries is also relevant. We develop a ranking based on weak expectations that satisfies the minimal value axiom.

II. One-stage and two-stage lotteries

It is commonly assumed that multi-stage lotteries (for example, 10% chance of L and 90% chance of L', where L and L' are themselves lotteries) are identical to, or at least equally valuable to the one-stage lottery with the same probabilities of outcomes. In this section, we make this idea precise and we formulate a framework in which the equivalence does not necessarily hold. For simplicity, we limit our attention to one-stage and two-stage lotteries.

We aim to define a binary relation $\succsim$ of *at-least-as-good,* defined on the collection (introduced below) of one- and two-stage lotteries, that extends the ranking of one-stage lotteries. The relation $\succsim$ is assumed to be transitive and reflexive. We do not assume that the relation $\succsim$ is complete. As usual, lottery L is as good as lottery L' just in case $L \succsim L'$ and $L' \succsim L$; and L is better than L' in case $L \succsim L'$ and not $L' \succsim L$.

A *one-stage lottery* is a sequence $X = ((p_1, v_1),(p_2, v_2),\ldots)$ of couples (p_k, v_k) of real numbers with p_k a positive probability and v_k a value. We assume throughout that outcomes have cardinal values that represent that which the agent seeks to promote. The probabilities sum to one. The order in which these couples are listed is not important: the sequence X and its permutations are considered identical lotteries. For convenience, the one-stage lotteries that we introduce below are listed in order of non-increasing probabilities, $p_1 \geq p_2 \geq \ldots$. If the sequence X is finite (resp. infinite), the one-stage lottery is said to be *finite* (resp. *infinite*). We make the convention that '$(p,v) \in X$' means that (p,v) is a component of the one-stage lottery X.

A *two-stage lottery* is a sequence $S = ((p_1, V_1),(p_2, V_2),\ldots)$ of couples (p_k, V_k) with p_1, p_2, … positive probabilities and V_1, V_2, … one-stage lotteries. In the first stage, a lottery V_n is realized with probability p_n. In the second stage, a prize is obtained according to V_n. Again, the probabilities sum to one and the order in which the (p_k, V_k)-couples are listed is not important. We will list them in non-increasing probabilities. The examples that we introduce below have *finite* second-stage lotteries. This is merely to simplify the presentation. Our results do not depend on this.

Let $\mathcal{L}$ collect all one- and two-stage lotteries. We shall refer to an element of $\mathcal{L}$ as a lottery (without further specification). If a lottery L and the certain one-stage lottery $((1,a))$ are equally good, then a is said to be the *certainty equivalent* or the *value* of L.

III. Three axioms

This section introduces the three axioms we impose: the weak-expectation axiom, the reduction axiom, and the minimal value axiom. In the next section we will show that these axioms are incompatible.

In order to introduce the first axiom, we recall the notion of weak expected value.[4] Let X be a one-stage lottery. The *standard* expected value of X (when it exists and is finite) is the value $E[X]$ to which the average value converges with probability one, as the sample size goes to infinity. That is, a long sequence of plays at any price above (resp. below) $E[X]$ will almost certainly lead to a loss (resp. a gain). The *weak* expected value $WE[X]$ of X is the value (when it exists) to which the average value converges in probability, as the sample size goes to infinity. That is, by fixing in advance a high enough number of plays, the average payoff per play can be almost guaranteed to be arbitrarily close to $WE[X]$. The weak expected value equals the standard expected value, when the latter exists. The former, however, can exist, when the latter does not (for example, the Pasadena game, as Easwaran shows).[5]

The weak expected value of a one-stage lottery can be determined as follows. Let the one-stage lottery X be weakly integrable, that is, (i) the lottery has *thin tails* in the sense that the product of the payoff n and the probability of the absolute value of payoffs being greater than n (that is, $n \times \text{Prob}(|v| > n)$), converges to zero as n goes to infinity, and (ii) the sequence $E[X^n]$ of

[4] Easwaran, "Strong and Weak Expectations," *op. cit.*.

[5] For related issues, see Luc Lauwers and Peter Vallentyne, "Decision Theory Without Finite Standard Expected Value," *Economics and Philosophy*, forthcoming, DOI: http://dx.doi.org/10.1017/S0266267115000334.

truncated expectations converges. Here, the truncated expectation $E[X^n]$ is the standard expected value of the truncated lottery X^n, where the latter is X but with all absolute payoffs greater than n replaced by payoffs of 0. (More exactly, X^n is the lottery that consists of all (p,v)-couples in the one-stage lottery X for which the absolute value of v, $|v|$, is less than or equal to n plus the couple $(q,0)$, where q equals the sum of the probabilities of the (p,v)-couples in X for which $|v| > n$.) The weak expected value, $WE[X]$, of the one-stage lottery X is equal to the limit of $E[X^n]$ as n goes to infinity. If the lottery is not weakly integrable, then it does not have a finite weak expected value (although it might have an infinite value).

The weak expected value of a one-stage lottery is determined on the basis of its distribution: if the one-stage lotteries X and Y have identical distributions and if X has a weak expected value, then also Y has a weak expected value and $WE[Y] = WE[X]$.[6]

Our first axiom requires that the relation $\succsim$, when restricted to the set of one-stage lotteries, agrees with the weak-expectation based ranking of one-stage lotteries.

Weak-expectation axiom. Let X and Y be two one-stage lotteries with weak-expected values. Then, X is at least as good as Y if and only if $WE[Y] \geq WE[X]$.

[6] Consider, for example, the one-stage lotteries $X = ((1,v))$ and $Y = ((.7,v),(.3,v))$. In principle, one could care not just about outcomes but also about states of affairs leading to those outcomes, in which case the agent might be non-indifferent between *X* and *Y*. In contrast, weak-expectation decision theory imposes indifference since *X* and *Y* have the same standard expected value.

In words, the relation $\succsim$ on $\mathcal{L}$ agrees with, and aims to extend, weak-expectation decision theory. Furthermore, the weak-expectation axiom entails that the certainty equivalent of the one-stage lottery X, if it exists, is equal to *WEV*[X].

We now introduce our second axiom. A common thought is that lotteries can be evaluated solely on the basis of their probability distributions for payoffs. This thought is captured by the reduction axiom below. Consider the two-stage lottery $S = ((p_i, V_i))_i$ with, for each *i*, the one-stage lottery V_i given by $((q_{i1}, v_{i1}),\ldots,(q_{ik_i}, v_{ik_i}))$. Figure 1 shows the lottery S by means of a tree.

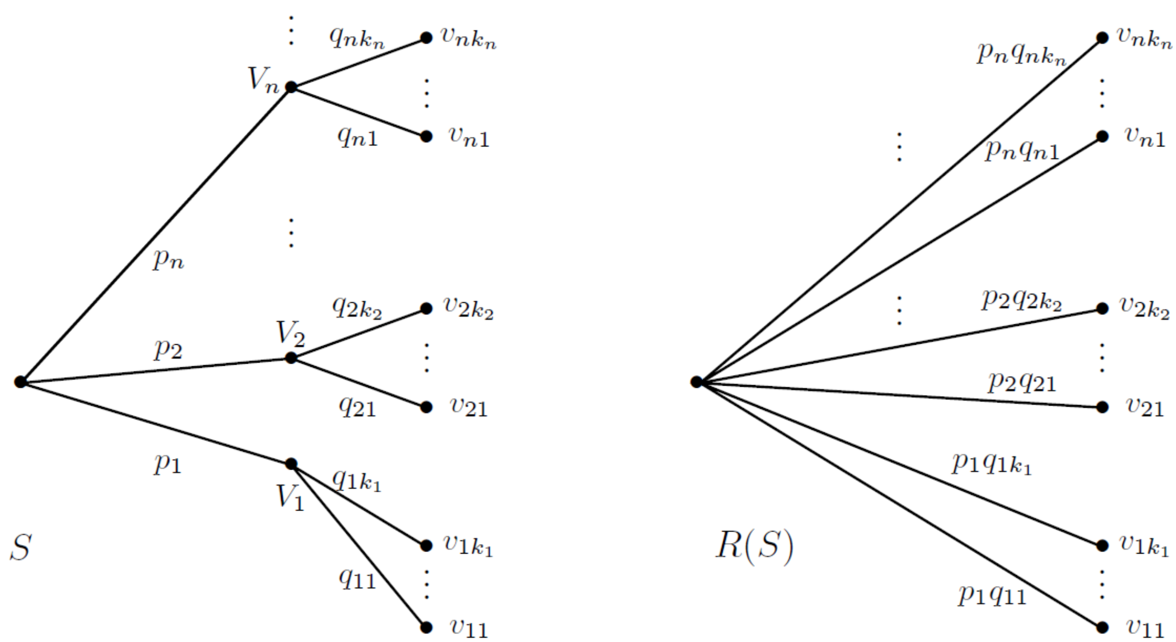

Figure 1. The two-stage lottery S and its reduced form $R(S)$.

The tree at the left represents the two-stage lottery S. Drawn from left to right, the tree starts with an initial node from which paths lead to the intermediate nodes that represent the outcomes of the first stage. The intermediate nodes correspond to the end-nodes of the first stage. The probability with which the first-stage lottery generates the lottery V_n as prize is written along the path that connects the initial node to the intermediate node V_n. From these intermediate nodes further branches are drawn towards the end-nodes which represent the prizes of the second-stage lotteries. The intermediate nodes are the initial nodes of the second-stage lotteries.

The tree at the right represents a one-stage lottery in which the value v_{ij} occurs with probability $p_i \times q_{ij}$, as it does in the two-stage lottery S. This one-stage lottery is said to be the *reduced form* of S, and we designate it $R(S)$. For each *i* and *j*, both lotteries, S and $R(S)$, realize the value v_{ij} with the same probability. According to the reduction axiom, the two-stage lottery S and the one-stage lottery $R(S)$ are equally good.

Reduction axiom. Let $S = ((p_1, V_1),(p_2, V_2),\ldots,(p_n, V_n),\ldots)$ be a two-stage lottery, with $V_i = ((q_{i1}, v_{i1}),$ $(q_{i2}, v_{i2}),\ldots,(q_{iki}, v_{iki}))$ a finite lottery for each *i*. Let $R(S)$, its one-stage reduced form, be

$$((p_1 q_{11}, v_{11}), (p_1 q_{12}, v_{12}), \ldots ,(p_1 q_{1k1}, v_{1k1}) ; (p_2 q_{21}, v_{21}), (p_2 q_{22}, v_{22}), \ldots ,(p_2 q_{2k2}, v_{2k2}) ; \ldots ;$$

$$(p_n q_{n1}, v_{n1}), (p_n q_{n2}, v_{n2}), \ldots ,(p_n q_{nkn}, v_{nkn}) ; \ldots).$$

Then, S and $R(S)$ are equally good.[7]

[7] The one-stage lottery $R(S)$ is not necessarily listed in non-increasing probabilities.

Each prize of the reduced form $R(S)$ corresponds to a prize of the two-stage lottery S, and it has the compound probability that the classical algebra of probability defines. As such, the reduction axiom turns a two-stage lottery into a one-stage lottery and imposes indifference. Whether the one-stage lottery $R(S)$ or the two-stage lottery S is behind the ultimate outcomes is, according to the reduction axiom, not significant. Accepting the reduction axiom entails that it is sufficient to rank one-stage lotteries.

Finally, we introduce our third axiom.

Minimal value axiom. Consider a two-stage lottery and a real number *a*. If each second-stage lottery is at least as good as the certain one-stage lottery $((1,a))$, then the two-stage lottery is not worse than $((1,a))$.

This principle seems sound. According to weak-expectation decision theory the value (or the certainty equivalent) of a two-stage lottery is the amount a risk-neutral agent is willing to pay in order to play the two-stage lottery. If (i) the two-stage lottery has a value, and (ii) each second-stage lottery has a value of at least *a*, then the agent is willing to pay (at least) the amount *a* to play the two-stage lottery.

IV. An impossibility result

This section presents the main result of this note.

Theorem. There does not exist a reflexive and transitive (but possibly incomplete) relation $\succsim$ on the set $\mathcal{L}$ that satisfies the weak-expectation axiom, the reduction axiom, and the minimal value axiom.

Proof (by contradiction): Assume the existence of a reflexive and transitive relation on $\mathcal{L}$ that satisfies the three axioms. We will define a two-stage lottery $U = ((p_n, K_n))_n$ the reduced form of which has a weak expected value equal to zero, but each K_i has a finite standard expected greater than .5. The reduced one-stage lottery $R(U)$ will have the following form $((2^{-(n+1)}, 2^{n+1}/(n+1)), (2^{-(n+1)}, -2^{n+1}/(n+1)))_{n=1,2,...}$. This one-stage lottery has thin tails and is symmetric. Therefore, the one-stage lottery $R(U)$ has weak expected value zero. Nonetheless, each K_i will have a standard expected value greater than .5.

Let us now define the second-stage lotteries, K_n, and show that they each have an expected value greater than .5. For each positive natural number *n*, the one-stage lottery K_n consists of three probability-value couples:

$$K_n = \left(\left(\frac{2^n}{3+2^n}, \frac{-2^{n+1}}{n+1}\right), \left(\frac{2}{3+2^n}, \frac{2^{2n}}{n2}\right), \left(\frac{1}{3+2^n}, \frac{2^{2n+1}}{2n+1}\right)\right).$$

The standard expected value of the lottery K_n is well defined and increases in n:

$$E[K_n] = \frac{2^{2n+1}}{3+2^n} \times \left(\frac{-1}{n+1} + \frac{1}{2n} + \frac{1}{2n+1}\right) > .5.$$

For example, for n equal to 1, 2, and 3 we have

$K_1 = ((.4,-2), (.4,2), (.2,8/3))$, $E[K_1] = .533$,

$K_2 = ((.571,-8/3), (.286,16/4), (.143,32/5))$, $E[K_2] = .533$,

$K_3 = ((.727,-4), (.182,64/6), (.091,128/7))$, $E[K_3] = .692$.

Now, plug in the probabilities $p_n = (3+2^n)/2^{2n+1} = (1+2+2^n)/2^{2n+1}$. The probabilities p_n add up to 1. Let us determine the reduced form $R(U)$. The final prizes are of the form $v_n = 2^{n+1}/(n+1)$ or $-v_n$. First, the prize $-v_n$ occurs in the lottery K_n. The compound probability reads $p_n \times 2^n/(3+2^n) = 2^{-n-1}$. Second, the prize v_n occurs in K_j with $j = n/2$ if n is even and $j = (n+1)/2$ if n is odd. The compound probability with which v_n is generated is (again) equal to 2^{-n-1}. In sum, the reduced one-stage lottery $R(U)$, written in non-decreasing absolute values, has the following form

$$\left(\left(\frac{1}{4}, 2\right), \left(\frac{1}{4}, -2\right), \left(\frac{1}{8}, \frac{8}{3}\right), \left(\frac{1}{8}, \frac{-8}{3}\right), \ldots, \left(\frac{1}{2^{n+1}}, \frac{2^{n+1}}{2n+1}\right), \left(\frac{1}{2^{n+1}}, \frac{-2^{n+1}}{2n+1}\right), \ldots\right).$$

This one-stage lottery is symmetric and has thin tails (that is, the product $n \times \text{Prob}(|v| > n)$ converges to zero as n goes to infinity). Therefore, the one-stage lottery $R(U)$ has weak expected value zero.

Hence, we end up with conflicting conclusions for the two-stage lottery U. On the one hand, the minimal value axiom entails that U cannot be ranked below .533. On the other hand, the reduction axiom and the weak-expectation axiom entail that U has value 0. And that entails that U is ranked below .533. Q.E.D.

So, at least one of the three axioms must be rejected. As the weak-expectation axiom is not part of standard decision theory, it is the most controversial one. As already mentioned, if this axiom is weakened to appeal to standard expected value, the conflict with the reduction axiom and the minimal value axiom disappears. Nevertheless, we think that weak-expectation decision theory is an important and promising approach to a more comprehensive evaluation of lotteries. Thus, we are inclined to accept weak-expectation decision theory (without further defense) and we investigate how it can be extended to multi-stage lotteries.

Consider next the minimal value axiom. In the context of weak expected decision theory, there is indeed some reason to be skeptical of the minimal value axiom. For standard expected value, the sum of the probability-weighted values is well defined in that it does not depend on the order in which the terms are added together. Thus, for a two-stage lottery, adding them together first for each second-stage lottery, and then adding those sums together, produces the same result as any other way of adding up the probability-weighted values. For two-stage lotteries with a weak expected value, but no standard expected value, however, the sum of the probability weighted values is not well defined. Thus, the above order of addition produces a result that is different from other orders of addition. One might therefore question whether it should be privileged the way that the minimal value axiom requires.[8] Nonetheless, the axiom has great intuitive appeal (further illustrated below) and thus we shall explore the rejection of the reduction axiom.

V. Reduction-based versus pruned-based ranking rules

This section presents two rules for ranking lotteries. One satisfies the weak-expected value axiom and the reduction axiom but violates the minimal value axiom. The other satisfies the weak-expected value axiom and the minimal value axiom but violates the reduction axiom.

To start, consider the following, where L and L' are two lotteries in $\mathcal{L}$, and the reduced form of a one-stage lottery is simply the one-stage lottery:

Reduction-based rule. $L \succsim_R L'$ if either $R(L) = R(L')$ or $WEV[R(L)] \geq WEV[R(L')]$.

This rule evaluates lotteries on the basis of their reduced forms. One lottery is at least as valuable as another if (1) they have the same reduced form (even if the reduced form has no weak expected value) or (2) both reduced forms have weak expected values and the first is at

[8] Indeed, one of us (Vallentyne) rejects the minimal value axiom for this reason.

least as great as the second. This rule satisfies the weak expected value axiom, the reduction axiom, but not the minimal value axiom (see below).

Let us now introduce a rule that satisfies the minimum value axiom. Instead of evaluating lotteries on the basis of their reduced forms, it evaluates them on the basis of their pruned forms, where this involves replacing second-stage lotteries having a finite weak expected value with that value.

More formally: Let $S = ((p_n, X_n))_n$ be a two-stage lottery. The *pruned form* of S, denoted by $P(S)$, is the lottery obtained from S by replacing each second-stage lottery having a finite weak expected value with a payoff equal to this value. Second-stage lotteries with no finite weak expected value are not altered. Let $C(X)$ denote, for each one-stage lottery X, the finite weak expected value of X, if it exists, or just X, if it does not. Then, the pruned form of the two-stage lottery S is given by $P(S) = ((p_n, C(X_n)))_n$. If each second-stage lottery has a finite weak expected value, the pruned form $P(S)$ is a one-stage lottery.

Consider, then, the following ranking rule, where L and L' are two lotteries in $\mathcal{L}$, and the pruned form of a one-stage lottery is simply the one-stage lottery:

Pruned-based rule. $L \succsim_P L'$ if either $P(L) = P(L')$ or $WEV[P(L)] \geq WEV[P(L')]$.

This rule evaluates lotteries on the basis of their pruned forms (rather than their reduced forms). One lottery is at least as valuable as another if they have the same pruned form (even if the pruned form has no weak expected value) or if both pruned forms are one-stage lotteries, have weak expected values, and the first is at least as great as the second. This rule satisfies the weak expected value axiom, the minimal value axiom (see below), but not the reduction axiom.

The reduction-based rule judges two-stage lotteries solely on the basis of their outcomes and their compound probabilities, with no special treatment to the probabilities of second-stage lotteries. In contrast, the pruned-based rule gives special treatment to the probabilities of second-stage lotteries. It sums probability-weighted value internal to second-stage lotteries, before summing those values over the different second-stage lotteries.

To illustrate the content of these rules, let us apply them to rank the two-stage lottery U = $\big((p_n, K_n)\big)_n$ introduced in the previous section against 0 for sure. As shown in the previous section, $R(U)$ has weak expected value of zero and thus the reduction-based rule considers U and $\big((1,0)\big)$ equally good. By contrast, the pruned-based rule considers U better than $\big((1,0)\big)$. The second-stage lotteries K_n are finite and have a certainty equivalent equal to their standard expected value $E[K_n]$. The pruned form, given by the one-stage lottery $P(U) = \big((p_n, E[K_n])\big)_n$ with $p_n = (3+2^n)/2^{2n+1}$, has standard expected value equal to

$$E[P(U)] = \sum_{n\geq 1} \frac{3+2^n}{2^{2n+1}} E[K_n] = \sum_{n\geq 1} \left(\frac{-1}{n+1} + \frac{1}{2n} + \frac{1}{2n+1} \right) = \log(2) \geq .533.$$

Thus, the two-stage lottery U has a pruned-based expected value of log(2).

Although the previous example reveals a difference, the two rules have a lot in common. Indeed, both rules extend standard expectation decision theory to cover one-stage lotteries with a weak expected value but no standard expected value. Furthermore, they both assess a single-stage lottery, $X = ((p_n, v_n))_n$, as equally valuable with both (1) the *degenerate* two-stage lottery, $((1, X))$, which has a 100% chance of X in the first stage, and (2) the *degenerate* two-stage lottery, $((p_n, ((1, v_n)))_n$, in which the *n*-th second stage lottery has probability p_n and, with certainty has a payoff of v_n.

Which rule is more plausible? This is a difficult issue, but below we note that the pruned-based rule has the attractive feature of satisfying a seemingly plausible dominance condition.

Two-stage dominance axiom. Let $S = ((p_n, X_n))_n$ be a two-stage lottery. Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be a sequence of one-stage lotteries such that $Y_n \succsim X_n$ for each *n*. Let $V = ((p_n, Y_n))_n$ be the two-stage lottery obtained from S by replacing X_n by Y_n for each *n*. Then $V \succsim S$.

The dominance axiom seems sound. If in the two-stage lottery S each second-stage lottery is replaced by a lottery that is at least as good, then the resulting two-stage lottery V is at least as good as S. The pruned-based rule satisfies this condition, since such replacements equal or increase the payoffs of the second-stage lotteries. By contrast, the reduction-based rule violates this condition, since, U, for example, has a reduction-based value of 0, but replacing each of its second-stage lotteries with their standard expected values (which are each greater than .5) produces a lottery of value log(2). Thus, the replacement of lotteries with equally valuable lotteries produces a lottery that is more valuable, in violation of the two-stage dominance axiom.

VI. Conclusion

Two-stage lotteries are lotteries that have other (one-stage) lotteries as prizes. If each lottery has only a finite number of end-nodes, a risk neutral agent is able to judge a two-stage lottery by means of its expected value. On the full set of one-stage and two-stage lotteries, however, three basic axioms --- the weak-expectation axiom, the reduction axiom, and the minimal value axiom --- become incompatible. We believe that the weak-expectation axiom is highly plausible (although not uncontroversial) and assume it throughout. The reduction axiom then entails the existence of a two-stage lottery the value of which is well defined and smaller than the smallest expected value of the second-stage lotteries. As the value of a lottery should be some average of the prizes involved, we reject the reduction axiom. We proposed an alternative route to evaluate (and to rank) infinite two-stage lotteries. This new criterion satisfies the minimal expected value axiom and violates the reduction axiom.