DECISION THEORY WITHOUT FINITE STANDARD EXPECTED VALUE

Luc Lauwers

KU Leuven

Faculty of Economics and Business

Naamsestraat 69 - bus 3565

3000 Leuven

Belgium

Luc.Lauwers@kuleuven.be

URL: http://feb.kuleuven.be/luc.lauwers

Peter Vallentyne

University of Missouri

406 Strickland Hall

Columbia, MO 65211-4160

U.S.A.

vallentynep@missouri.edu

URL: http://Klinechair.missouri.edu

Abstract:

We address the question, in decision theory, of how the value of risky options (gambles) should be assessed when they have no finite standard expected value, that is, where the sum of the probability-weighted payoffs is infinite or not well defined. We endorse, combine, and extend (1) the proposal of Easwaran (2008) to evaluate options on the basis of their weak expected value, and (2) the proposal of Colyvan (2008) to rank options on the basis of their relative expected value.

Keywords: Pasadena paradox, expected value, weak expectations, relative expectations

We address the question, in decision theory, of how the value of risky options (gambles) should be assessed when they have no finite standard expected value, that is, where the sum of the probability-weighted payoffs is infinite or not well defined. We endorse, combine, and extend (1) the proposal of Easwaran (2008) to evaluate options on the basis of their weak expected value, and (2) the proposal of Colyvan (2008) to rank options on the basis of their relative expected value.

Our goal is to outline a framework rather than to give a compelling defense of it. We shall motivate, through the use of examples, the plausibility of principles that go beyond standard expected value.

1. The Problem

We address the question in decision-theory of how (risky) options should be evaluated when they have no finite standard expected value. In this section, we shall define these terms and explain the problem.

Throughout, we restrict our attention to cases where an option determines a countable set $\{<p_1,v_1>, <p_2,v_2>, \ldots, <p_k,v_k>, \ldots\}$, where p_k is the probability of receiving a payoff of v_k. We thus do not address cases where there are uncountably-many distinct finite payoffs with non-zero probability. Nor do we address cases where probabilities or payoffs are undefined, involve infinitesimals, or are infinite. We further assume that the values of outcomes are independently specified (e.g., monetary values or wellbeing values) on an interval scale. Thus, we assume that there is a fact about whether the difference in value between two outcomes is less, equal, or greater than the difference in value between two other outcomes. We leave open what kind of value (e.g., prudential vs. moral) is used, since that may vary based on the nature of the decision problem.

We further assume that the evaluation of options is risk neutral. Thus, we assume that, in the standard cases, risky options are evaluated on the basis of their standard expected value, that is, on the basis of the probability-weighted sum $p_1v_1 + p_2v_2 + \ldots + p_nv_n + \ldots$. The problem, as we shall explain, is that sometimes this sum is not well defined. The usual assumption is that such options cannot be evaluated. We shall endorse principles proposed by Easwaran (2008) and Colyvan (2008) and then combine and extend those principles.

The sum of a set of numbers is well-defined if and only if the order of summation does not matter, that is, if and only if each order of summation results in the same total. A crucial mathematical fact is that the *sum of a (countable) set of numbers is well defined* if and only if the sum of the positive terms is finite (or there are no such terms) or the sum of the negative terms is finite (or there are no such terms). If both sums are finite, the sum of the entire set is simply the sum of these two finite numbers. If one sum is finite, and the other sum is (positively or negatively) infinite, the sum of the entire set is well defined and correspondingly infinite. If both sums are infinite, then there is no well-defined sum. For such sets the order of summation matters.

For example, for the set {…, –1/32, –1/8, –1/2, 1, 1/4, 1/16, …}, the sum of the positive numbers (1+1/4+1/16+…) is 4/3 and the sum of the negative numbers (–1/2–1/8–1/32–…) is –2/3. This establishes that the above set of values has a well-defined sum equal to 2/3 (=4/3–2/3).

For the set {–1/2, 1, 1/3, 1/5, …, 1/(2*n*+1), …}, the sum of the negatives is finite (–1/2) and the sum of the positives is infinite. Hence, this set has a well-defined sum, which is positive infinity. By contrast, for the set { …, –1/(2*n*),…, –1/6, –1/4, –1/2, 1, 1/3, 1/5, …, 1/(2*n*+1), …}, the sum of the negatives is infinite, as is the sum of the positives, and hence this set has no well-defined sum.

Consider now the Pasadena game, introduced by Nover and Hájek (2004).[1] A fair coin is flipped until a heads comes up, and one wins something if the number of flips is odd and loses something if the number of flips is even. More precisely, the payoffs, along with the associated probabilities are described as follows:

Pasadena

Positive payoffs: $< 1/2, 2/1>, < 1/8, 8/3>, \ldots, < 1/2^{2n-1}, 2^{2n-1}/(2n-1)>, \ldots$

Negative payoffs: $< 1/4, -4/2>, < 1/16, -16/4>, \ldots, < 1/2^{2n}, -2^{2n}/2n>, \ldots$

The set of probability-weighted payoffs is thus {…, –1/2*n*, …, –1/6, –1/4, –1/2, 1, 1/3, 1/5, …, 1/(2*n*–1), …}. Given that the sum of the negatives is infinite and the sum of the positives is infinite, this set has no well-defined sum. Nonetheless, it seems natural to add these terms in the order of how many flips it takes for a heads to occur and thereby realize the payoff: 1 –1/2 + 1/3 –1/4 …+1/(2*n*–1) –1/(2*n*) …. Added in this order, the sum is log(2), or approximately .69. The problem is that adding the terms in a different order can give a different sum. For example, adding the very same terms above in the following order (1+1/3–1/2)+(1/5+1/7–1/4)+(1/9+1/11–

[1] See also, Hájek and Nover (2006), Hájek and Nover (2008), and Hájek (2009).

1/6)+ … produces a total of 1.5 log(2), or approximately 1.04. The sum is not well-defined precisely because there is no order-independent fact about what the terms sum to.

The general problem that we address is the evaluation of risky options for which there is no standard expected value, defined as follows:

Standard Expected Value. The *standard expected value* of an option exists and has value *v* (which can be infinite) if and only if the sum of the (countable many) probability-weighted payoffs is well defined and equals value *v*.

Throughout, we allow that the value of an option can be infinite. Consider, for example, the well-known St. Petersburg game, where a fair coin is flipped until it lands on heads, and, if that takes *n* flips, the payoff is 2^n. This determines the probability-payoff set {<1/2, 2>, <1/4, 4>, …, $< 1/2^n, 2^n >$, …}. This has a standard expected value equal to positive infinity, since the sum of the probability-weighted positive payoffs is infinite, and there are no negative payoffs.

It is important to remember that having an infinitely positive value only establishes that the option is more valuable than any option with a finite value. Two options with infinite positive value need not be equally valuable. For example, if the payoffs of the St. Petersburg game are increased by one unit, the result is an infinitely valuable option that is arguably more valuable than the original, and certainly not equally valuable with it.

Throughout, we shall assume the following relatively uncontroversial claim, given our assumption of risk-neutrality:

Standard Expectations. The value of an option is its standard (finite or infinite) expected value, if it exists.

We shall now formulate and endorse some principles of evaluation when options have no standard expected value. More specifically, we shall combine and extend the weak expectations principle advocated by Easwaran (2008) and the relative expectations principle advocated by Colyvan (2008).

2. Weak and Strong Expectations

Kenny Easwaran (2008) has tentatively suggested that the Pasadena game (above) has a value of log(2) on the following basis: (1) He distinguishes between weak expected value and strong expected value (defined below). (2) He establishes that the Pasadena game has a weak expected value of log(2). (3) He tentatively suggests that, in general, risky options can be assessed on the basis of their weak expected value.

Consider an option, X, and let $E(X)$ be its standard expected value. Easwaran appeals to two versions of the law of large numbers to define weak and strong expected value for an option. Let $Ave(X,n)$ be the average value of X for n independent trials. The two laws are:

Strong Law of Large Numbers. For any option, X, for which $E(X)$ exists, there is a probability of 1 that the limit, as n goes to infinity, of $|Ave(X,n)–E(X)|$ is 0.

Weak Law of Large Numbers. For any option, X, for which $E(X)$ exists, for any positive number, e, the limit, as n goes to infinity, of the probability that $|Ave(X,n)–E(X)|<e$ is 1.

Both laws concern the probability of the standard expected value and the sample average value being arbitrarily close to each other. The difference between the two laws concerns whether the limit, as the sample size goes to infinity, is internal to the probability assignment (the strong law) or external to it (the weak law).

These two laws define the weak and strong expected value as follows:

Finite Strong Expected Value. The *strong expected value* of an option, *X*, exists and has finite value *v* if and only if there exists a real number *v* such that, with probability 1 the limit, as *n* goes to infinity, of |Ave(*X,n*)–*v*| is 0.

Finite Weak Expected Value. The *weak expected value* of an option, *X*, exists and has finite value *v* if and only if there exists a real number *v* such that, for each positive number *e*, the limit, as *n* goes to infinity, of the probability that |Ave(*X,n*)–*v*|<*e* is 1.

In each of these definitions, the stochastic behavior of the sample average, Ave(*X,n*), is used to define a value for *X*. The strong expected value of *X* is the value *v* for which there is *a probability of 1 that, for large enough sample sizes,* the average value Ave(*X,n*) will be arbitrarily close to *v*. The weak expected value, by contrast, is the value *v* for which, *for large enough sample sizes, there is a probability arbitrarily close to 1* that the average value will be arbitrarily close to *v*. Whenever the strong expected value is defined, the weak expected value is also defined and has the same value.

It turns out that, in the finite case, strong expected value just is standard expected value (although we shall see that this is not so in the infinite case):

Finite Strong Expected Value Lemma (based on Durrett 2005: Ch. 1, sec. 8)**.** An option has a finite strong expected value of *v* if and only if it has a standard expected value of *v*.

Thus, the following is equivalent to the uncontroversial Finite Standard Expectations:

Finite Strong Expectations. The value of an option is its strong expected value, if it exists and is finite.

Because Pasadena has no finite standard expected value, it has no finite strong expected value. Although the existence of a finite strong expected value entails the existence of a finite weak expected value, the reverse entailment does not hold. Indeed, as Easwaran shows, drawing on Feller (1971) and Durrett (2005), Pasadena has a weak expected value of log(2), even though it has no finite strong expected value.

More generally, a necessary and sufficient condition for the existence of finite weak expected value is provided by the following lemma, where the terms are defined as follows. Let $\Pr(|X|>x)$ be the probability that the absolute value of the payoff of option *X* is greater than the real number *x*. Say that an option has *thin tails* if and only if *x* multiplied by $\Pr(|X|>x)$ converges to zero as *x* goes to positive infinity. Finally, let X_n be an option with the same payoffs as *X*, except that all payoffs with absolute values above *n* are set equal to zero. For example, for Pasadena, *X*, X_2 sets all absolute payoffs above 2 equal to 0 and thus its probability-payoff pairs are <1/2, 2>, <1/4, –4/2>, and <1/2*n*, 0>, for *n*>2. The standard expected value of X_n, $E(X_n)$, is always well defined, given that truncated payoffs are bounded.

Finite Weak Expected Value Lemma (Feller 1971, chapter VII, Thm. 1 and Durrett 2005, Chapter 1, 5.5) **.** Option X has a finite weak expected value, v, if and only if (1) X has *thin tails*, and (2) the limit of $E(X_n)$, as n goes to positive infinity, is v.

The first condition requires that the limit of x multiplied by $\Pr(|X|>x)$ go to zero, as x goes to infinity. For example, for Pasadena, $\Pr(|X|>2) = 1/4$ (since $\Pr(X=2)=1/2$ and $\Pr(X=-4/2) = 1/4$, and all other payoffs have an absolute value greater than 2), and 2 multiplied by $\Pr(|X|>2)$ is 2 multiplied by 1/4, or 1/2.

To determine the weak expected value on the basis of this lemma, for each payoff level n, one truncates the option X at level n, determines the strong expected value of this truncated variable, and then takes the limit as n goes to infinity. Although the Pasadena game has no standard expected value, it has a weak expected value of log(2). To see this, note that the absolute payoffs of the Pasadena have the form $2^n/n$, and thus condition (1) is equivalent to the requirement that $(2^n/n)\Pr(|X|>2^n/n)$ converge to zero, as n goes to infinity. For Pasadena, $\Pr(|X|>2^n/n) = 1/2^n$, and thus (1) requires that $(2^n/n)(1/2^n)$, or $1/n$, converge to zero as n goes to infinity. Hence, condition (1) of the lemma is satisfied. To see that condition (2) also is met, note that $E(X_n) = 1-1/2+1/3+\ldots+ (-1)^n /n$.[2] Thus, the limit of $E(X_n)$ is log(2) as n goes to infinity. Hence, condition (2) of the definition is met, and Pasadena has a weak expected value equal to log(2). It is worth noting that the lemma ensures that the weak expected value is based on the summation of probability-weighted payoffs in increasing order of the absolute value of the

[2] See, for example, http://en.wikipedia.org/wiki/Alternating_harmonic_series.

payoff. This is in contrast to Nover and Hájek (2004), who argue that no particular order of summation is privileged.

If a risky option has a finite standard expected value, then it has a finite weak expected value, and both values coincide. This follows from the fact that the standard expected value is finite only if the sum of the probability-weighted payoffs is finite for both the positive, and the negative, payoffs. That in turn implies that (1) X has thin tails and (2) the sum of the probability-weighted payoffs does not depend upon the order of summation (which ensures the sum in increasing order of the absolute value of the payoffs gives the correct answer).Thus, the concept of weak expected value is a strengthening of standard expected value.

Easwaran tentatively proposes that Finite Strong Expectations be strengthened to the following principle:

Finite Weak Expectations. The value of an option is its weak expected value, if it exists and is finite.

As Easwaran notes, a player who plays a game a very large number of times at a price that is slightly higher (respectively: lower) than the weak expectation has a very high probability of ending up behind (respectively: ahead). Indeed, by repeating the game enough times, that probability can be made as close to 100% as one likes.

Although Finite Weak Expectations is not uncontroversial,[3] we find it compelling.[4] In the

---

[3] For additional discussion of Easwaran's approach, see Fine (2008), Sprenger and Heesen (2009), and Smith (2014).

remainder of the paper we shall strengthen it in various ways.

3. Infinite Expectations

In this section we extend Finite Weak Expectations to cover cases where there is infinite weak expected value.

Consider the following option:

Squared St. Petersburg–Pasadena

Positive payoffs: $< 1/2, 4>, <1/8, 2^{6}>, <1/32, 2^{10}>, \ldots, <1/2^{2n-1}, 2^{4n-2}>, \ldots$

Negative payoffs: $<1/4, -2>, <1/16, -4>, <1/64, -64/6>, \ldots, <1/2^{2n}, -2^{2n}/2n>, \ldots$

Here, for a given probability, the positive payoffs are the squares of the St. Petersburg payoffs for those probabilities, and the negative payoffs, are the same as the payoffs of Pasadena for those probabilities. Both the positive and the negative parts have infinite totals. Hence, there is no standard expected value. Nevertheless, this option has infinite strong expected value, defined as follows:

Infinite Strong Expected Value. Option ***X*** has infinitely positive (respectively: negative) strong

[4] Actually, only one of us (Vallentyne) finds Finite Weak Expectations compelling. The other (Lauwers) finds it compelling for one-stage lotteries but not for compound lotteries (e.g., involving sequential coin flips). We discuss the problems for multi-stage lotteries in Lauwers and Vallentyne (2015).

expected value if and only if, for each positive number, e, with probability 1, for all sufficiently large n, Ave(X,n)$>e$ (resp. $<-e$).

This is the same as the definition of *finite* strong expected value, except that it requires that Ave(X,n) become arbitrarily large in absolute value rather than arbitrarily close to some finite value.

An application of the theorem of Derman and Robbins (1955) establishes that the above option has an infinite strong expected value, even though it has no standard expected value.

The crucial point here is that, although finite standard expected value and finite strong expected value always are the same, *infinite* strong expected value can exist when there is no standard expected value (but not vice-versa).

We believe that the following strengthening of Standard Expectations is plausible:

Strong Expectations. The value of an option is its strong (finite or infinite) expected value, if it exists.

Indeed, we believe that a further strengthening is plausible. Consider:

Weakly Infinite Pasadena

Positive payoffs: $<1/2, (2/1)1.01>$, $<1/8, (8/3)1.01>$, $<1/32, (32/5)1.01>$, …,

$<1/2^{2n-1}, (2^{2n-1})1.01/(2n-1)>$, …

Negative payoffs: $<1/4, -4/2>$, $<1/16, -16/4>$, $<1/64, -64/6>$, …, $<1/2^{2n}, -2^{2n}/2n>$, …

This is the same as Pasadena, except that the positive payoffs are multiplied by 1.01.

Corollary 1 of Erickson (1973) establishes that the above option has no strong expected value. Showing this, however, would be complex, and we omit that demonstration.[5]

Although Weakly Infinite Pasadena has no strong expected value, it has infinite weak expected value, defined as follows:

Infinite Weak Expected Value. Option ***X*** has positive (respectively: negative) infinite weak expected value if and only if, for each positive number, *e*, and each positive number, *d*, strictly between 0 and 1, for all sufficiently large *n*, the probability that Ave(*X*,*n*)> *e* (resp. < –*e*) >1–*d*.

This is the same as the definition of *finite* weak expected value, except that it requires that Ave(*X*,*n*) become arbitrarily large in absolute value rather than arbitrarily close to some finite value. It is like the definition of infinite *strong* expected value, except that it requires that, for all sufficiently large sample sizes, the probability of the average being greater than any given value *converges to 1* rather than that there be *probability 1* that, for all sufficiently large sample sizes, the average is greater than any given value. Whenever the strong expected value (finite or infinite) exists, the weak expected value exists and has the same value. Weak expected value

---

[5] Roughly: For an option for which E(|*X*|) is infinite, the strong expected value is positively (respectively: negatively) infinite if and only if a certain measure, J_{-} (respectively: J_{+}) is finite. Given that Pasadena does not have an infinite strong expected value, both of these measures are infinite. Multiplying the positive payoffs by 1.01 leaves both measures infinite. Thus, Weakly Infinite Pasadena does not have infinite strong expected value either.

(finite or infinite), however, can exist without strong expected value existing.

To see that the above option has infinite weak expected value, we can appeal to the following lemma:

Infinite Weak Expected Value Lemma (Durrett 2005: Ch. 1, sec. 8)**.** An option, X, has a positively (respectively: negative) infinite weak expected value if: (1) X has *thin tails* (i.e., x multiplied by $\Pr(|X|>x)$ converges to zero as x goes to positive infinity), and (2) the limit of $E(X_n)$ is positively (resp: negatively) infinite.

This lemma is similar to the corresponding one for *finite* weak expected value, except (1) it requires that the limit of $E(X_n)$ be infinite, and (2) it supplies only a sufficient condition for weak expected value.[6]

The above option has thin tails and the limit of $E(X_n)$ is infinite. Thus, it has infinite weak expected value.

We believe that Strong Expectations can plausibly be strengthened to:

Weak Expectations. The value of an option is its weak (finite or infinite) expected value, if it exists.

Of course, those who reject Finite Weak Expectations will also reject the infinite version,

---

[6] An important additional sufficient condition for the existence of infinite weak expected value is given by Theorem 2 of Baum (1963).

as will those who reject infinite value even for standard expected value. We believe, nonetheless, that Weak Expectations is plausible and shall assume it below.

Not all options, of course, have a (finite or infinite) weak expected value. The following is an example of one that does not.

Symmetric St. Petersburg (SP*)

Positive payoffs: < 1/4, 4>, < 1/8, 8>, …, <$1/2^{n+1}$, 2^{n+1}>, …

Negative payoffs: < 1/4, –4>, < 1/8, –8>, …, <$1/2^{n+1}$, -2^{n+1}>, …

This is like the St. Petersburg, except that it has negative payoffs defined to be symmetric with the positive payoffs.

The Finite Weak Expected Value Lemma above establishes that this does not have *finite* weak expected value (because it does not have thin tails). Moreover, for each positive number, v, the probability of winning v is the same as the probability of losing v. It follows that, for any positive n and k, the probability that Ave(SP*,2^n) is larger than k is equal to the probability that Ave(SP*,2^n) is smaller than $-k$. Hence, the option SP* does not have an infinitely positive, or infinitely negative, weak expected value.

Weak Expectations makes comparative cardinal assessments when both options have a weak expected value, with at least one value being finite. If they both have finite weak expected value, say m and n respectively, then (1) they are equally valuable if m=n, and (2) the option with value m is m–n units more valuable than the other (where being –c units more valuable is being c units *less* valuable). If one option has positively infinite value, then it is infinitely more valuable than the other. If one option has negatively infinite value, then it is infinitely less valuable than the other. If both values are infinite, however, no comparative assessment follows. One option may be more valuable, less valuable, equally valuable, or incomparable with the other option

We shall now strengthen the principle to cover certain cases where at least one option has no weak expected value.

4. Indeterminate Expectations

The weak expected value of an option has finite value *k* if and only if the probability that its average is arbitrarily close to *k* converges to one, as the sample size goes to infinity. We shall now generalize this notion to include *interval* assessments of weak expected value (e.g., a value of between 2 and 6 units). This will enable comparative cardinal assessments in some cases where at least one of the options lacks a (precise) weak expected value.

To see the need for strengthening the weak expectations principle, consider the following option, O, which is the same as Pasadena except that the signs of the payoffs are adjusted to ensure that, for each n, $E(O_n)$ (the expected value of the n-truncation of O) is inclusively between 0 and 1. Thus, for example:

Oscillating Weak Expected Value

Probability:	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	…
O	2	–4/2	–8/3	16/4	32/5	64/6	128/7	–256/8	…

Option O has no (finite or infinite) weak expected value. Although O has thin tails, the sequence of the truncated expected values, $E(O_n)$, oscillates, as n goes to infinity, between 0 and 1. (The above specification of the signs of the payoffs ensures exactly this.) Thus, Weak

Expectations is silent. Nonetheless, it is plausible, we claim, that O has a value of at least 0 and at most 1. We shall suggest that it has an "interval value" of [0,1].

In order to introduce the concept of interval value, we need to first introduce the following two standard mathematical concepts. The *greatest lower bound* of a set of numbers is (a) the largest (finite) real number that is a lower bound for (i.e., smaller or equal to) all members of the set, if a lower bound exists, and (b) negative infinity, if there are no lower bounds. For example, 2 is the greatest lower bound for $\{2,3,4, \ldots, n, \ldots\}$, and $-\infty$ is the greatest lower bound for $\{\ldots, -n, \ldots, -4, -3, -2\}$. The *least upper bound* of a set of numbers is (a) the smallest (finite) real number that is an upper bound for (i.e., greater or equal to) all members of the set, if there is an upper bound, and (b) positive infinity, if there are no upper bounds. For example, –2 is the least upper bound for $\{\ldots, -n, \ldots, -4, -3, -2\}$, and ∞ is the least upper bound for $\{2,3,4, \ldots, n, \ldots\}$.

We can now define the weak expected *interval* value of an option:

Weak Expected Interval Value. The weak expected interval value of option X is the closed interval $[x_1,x_2]$, where: (1) x_1 is the greatest lower bound on the value of k for which lim pr(Ave(X,n)$\geq k$) = 1, and (2) x_2 is the smallest upper bound on the value of k for which lim pr(Ave(X,n)$\leq k$) = 1.

All options have a weak expected interval value. An option with finite weak expected value of k has an interval value of $[k,k]$, and an option with infinite weak expected value has an interval value of $[\infty,\infty]$. An option with a completely indeterminate weak expected value (e.g., Symmetric St. Petersburg, from the previous section) has an interval value of $[-\infty,\infty]$. Option O, above, has an interval value of [0,1]. It is thus worth more than any negative value and worth no more than 1 unit of value.

This provides the basis for the evaluation of options on the basis of the following plausible principle:

Ordinal Interval Weak Expectations. Option, X, with weak expected interval value $[x_1,x_2]$, is at least as valuable as option, Y, with weak expected interval value $[y_1,y_2]$, if $x_1 \geq y_2$.[7]

Here we stipulate, as is standard for the extended reals, that (1) for any finite number, n, $\infty > n > -\infty$, and (2) neither $\infty \geq \infty$, nor $-\infty \geq -\infty$.

Thus, if X's greatest lower bound is at least as great as Y's least upper bound, then X is at least as valuable as Y. For example, X with value [3,4] is at least as valuable as Y with value [2,3], but Y is not at least as valuable as X (since it is not the case that $2 \geq 4$). Of course, for some options, Ordinal Interval Weak Expectations is silent. For example, it makes no comparative assessment of W with value [3,10] against Z with value [4,6].

This principle can be plausibly strengthened to say *how much* more valuable one option is than another. Consider then the following principles where it is stipulated that (1) $\infty+n=\infty$, for n finite or ∞, (2) $-\infty+n=-\infty$, for n finite or $-\infty$, and (3) $\infty+-\infty$ is not defined:

Cardinal Interval Weak Expectations. For option, X, with weak expected interval value $[x_1,x_2]$, and option, Y, with weak expected interval value $[y_1,y_2]$, X is *not less than* (x_1-y_2) units, *and not*

[7] See Fishburn (1985) for more on interval orders.

more than (x_2–y_1) units, more valuable than *Y*.

This principle holds, for example, that, for *X* with weak expected interval value [4,5] and *Y* with weak expected interval value [2,3], *X* is not less than 1 unit more valuable than *Y* and not more than 3 units more valuable. Moreover, for *Z* with weak expected interval value of [2,6], *X* is at least –2 units more valuable (i.e., 2 units less valuable) than *Z* and not more than 3 units more valuable. That is, *X* is not determinately more valuable than Z, but there are limits on how much worse (–2) and how much better (3) it is.

It is important to keep in mind here that Cardinal Interval Weak Expectation sets lower and upper bounds on the difference in value between two options, but it does not claim that these bounds are the *greatest* lower bounds or the *least* upper bounds. Thus, for example, the claim that *X* is at least –2 units more valuable and at most 3 units more valuable than *Z* is compatible with the claim that *X* is exactly 1 unit more valuable than *Z* (but not with the claim that it is exactly 4 units more valuable). Indeed, many of the relatively indeterminate assessment of Cardinal Interval Weak Expectations will be strengthened to more precise assessment by the principles of the next section.

With respect to options with a weak expected value, Cardinal Interval Weak Expectation has the same implications as Weak Expectations. An option has a weak expected value if and only if its weak expected value interval has the form [*k*,*k*], for some *k* (finite or infinite). For any two such options, say with intervals [*k*,*k*] and [*l*,*l*] the first is *k*–*l* to *k*–*l* units at least as valuable as the second, which is to say *k*–*l* units more valuable, as required.

Cardinal Interval Weak Expectations makes, however, many comparative judgments that Weak Expectations does not. For example, as noted above, it judges *X*, with weak expected interval value [4,5], to be at least 1 unit, and not more than 3 units, more valuable than *Y*, with weak expected interval value [2,3], even though neither option has a weak expected value.

In the following sections, we examine some principles for addressing two kinds of cases for which Cardinal Interval Weak Expectations makes no non-empty comparative assessment. One kind of case is where the subtractions involved are not well defined. This occurs where the interval values are $[x_1,x_2]$ and $[y_1,y_2]$, where both x_1 and y_2, or both y_1 and x_2, are positively infinite, or both negatively infinite (e.g., $[-1,\infty]$ compared with $[\infty,\infty]$, or $[-\infty,1]$ compared with $[-\infty,-\infty]$. The second kind of case is where the assessment is radically indeterminate in the sense that the assessment is that one option is at least $-\infty$, and at most ∞, units more valuable than the other. This occurs where (1) both options have ∞ as their greatest interval value but not as their lowest interval value, or (2) both options have $-\infty$ as their lowest interval value but not as their highest interval value (e.g., $[2,\infty]$ compared with $[1,\infty]$). As we shall see, some additional principles make comparative assessments possible in some of these cases.

5. An Interlude on State Spaces

The weak, strong, and standard expected values of an option are determined by its payoff distribution, $\{<p_1,v_1>, <p_2,v_2>, \ldots, <p_k,v_k>, \ldots\}$, where p_k is the probability of receiving a payoff of v_k. Options, however, have more structure than this, and the principles that we introduce below (unlike those introduced above) depend on this structure. We shall now make that structure explicit.

Options are defined over state spaces. A *state space* is a set of basic states, along with an associated probability function over certain specified subsets of those states (the events). Because we restrict our attention to *countable* state spaces (i.e., with countably-many basic states) and *countably additive* probability functions (i.e., the probability of a union of a countable number of basic states is the sum of their individual probabilities), a state space can be represented by a countable set of couples $\{<s_1,p_1>, <s_2,p_2>, \ldots, <s_i,p_i>, \ldots\}$, where s_i is a basic state, p_i is the positive probability with which s_i occurs, and the p_i sum to one. (Throughout, we assume that the probabilities of states are independent of the options chosen.)

An option can be represented as a function that assigns an outcome to each basic state (of its associated state space). As is standard, we assume throughout that options defined on the same state-space are equally valuable, if, for each basic state, they have equally valuable outcomes (even if the outcomes are distinct). Thus, we can represent options as real-valued functions that assign to each basic state the value of its outcome for the state, expressed in units of the relevant value (here left open). An option can thus be represented by a countable set of triples $\{<s_1,p_1,v_1>, <s_2,p_2,v_2>, \ldots, <s_i,p_i,v_i>, \ldots\}$, with s_i and p_i as above, and with v_i the value (or payoff) of the outcome of the option under state s_i.

Above, in discussing weak and strong expected value, we simplified this by combining states with the same payoff (and adding together the associated probabilities). The remaining principles, however, require the fuller state-space specification.

Two options defined on different state spaces can always be represented on a common state-space. If X is defined on states s_1, s_2, … s_n, …, and Y is defined on states r_1, r_2, …, r_n, then their common space is simply the set $\{s_1\&r_1, s_2\&r_1, \ldots, s_n\&r_1, \ldots, s_1\&r_2, s_2\&r_2, \ldots s_n\&r_2, \ldots, s_1\&r_n, s_2\&r_n, \ldots, s_n\&r_n, \ldots\}$, where $pr(s_n\&r_m) = pr(s_n)pr(r_m|s_n)$. If the states are interdependent (e.g., r_n=~s_i), then some of these combinations are impossible, have probability zero, and ignored/eliminated.

We shall now address some additional principles for assessing options defined on the same state space.

6. Relative Expectations

If two options each have weak expectations, with at least one of them finite in value, then Weak Expectations determines how much more valuable one is compared with the other. If the two options do not have any weak expectation but each has a *weak expected interval value*, then Cardinal Interval Weak Expectations will typically give a non-trivial assessment of some lower and upper limits on how much more valuable one option is compared with the other (e.g., at least 2 and at most 4 units more valuable).

As indicated above, however, there are two cases where this is not so. One is where both options have positively infinite, or both have negatively infinite, weak expected value. In this case, Cardinal Interval Weak Expectations is silent, for example, because comparing $[\infty, \infty]$ with $[\infty, \infty]$ involves $\infty - \infty$ and that is undefined. The second kind of case is where both options have a weak expected interval values of the form $[n,\infty]$, or both have the form $[-\infty,n]$, where n can be finite or infinite and need not be the same for the two options. In this case, Cardinal Interval Weak Expectations judges each option to be $-\infty$ to ∞ units more valuable than the other. That is true but uninformative. We shall now add a principle, appealing to the state-space structure, that covers some (but not all) of these cases.

A very plausible principle, which we shall strengthen below, is:

Strong Dominance. If, (1) *X* and *Y* are options defined on the same state space, (2) for each basic state, *X* has a payoff that is at least as great as that of *Y*, and (3) for some basic states, *X* has a greater payoff than *Y*, then *X* is more valuable than *Y*.

Strong Dominance is highly plausible but very weak. Consider SP (St. Petersburg) and SP*:

Relative Expectations

Probability:	1/2	1/4	1/8	1/16	1/32	…	$1/2^n$	…
State:	s_1	s_2	s_3	s_4	s_5	…	s_n	…
SP	2	4	8	16	32	…	2^n	…
SP*	1.8	5	9	17	33	…	2^n+1	…
SP*–SP	–.2	1	1	1	1	…	1	…

For each state other than s_1, SP* has payoff that is one unit greater than SP, but for s_1 SP* has a payoff that is .2 lower. Thus, there is no strong dominance here. Nonetheless, it is extremely plausible that SP* is more valuable than SP.

The following principle from Colyvan (2008) captures this assessment, where *X*–*Y* is an option that, for each state, has an outcome whose value is equal to the value of *X*'s outcome minus the value of *Y*'s outcome. We assume that there is always at least one such outcome, and that, if there is more than one, then they are all equally valuable (since, for each basic state, their outcomes have the same value).

Relative Expectations (Sufficiency Version)**.** If *X*–*Y* has a standard expected value that is nonnegative, then *X* is at least as valuable as *Y*.[8]

SP and SP* each have infinite standard expected value. Nonetheless, the standard expectation of (SP*–SP) is positive (.4 = –.1 + .5 =1/2(–.2) + ¼(1) + 1/8(1) …). Thus, Relative Expectations rightly says that SP* is more valuable than SP.

Colyvan (2008) endorses a stronger version of this principle, which also holds that *X* is at least as valuable as *Y only if* the standard expected value of *X*–*Y* exists and is non-negative. We believe that this is too strong. We believe that the relative value of two options can be assessed in many cases where this condition fails. We show this below by showing that stronger versions of the principles are plausible.

First (as recognized by Colyvan and Hájek 2015), it is plausible to endorse a cardinal version of Relative Expectations:

Cardinal Relative Expectations. If the standard expected value of *X*–*Y* is *n* (which can be infinite), then *X* is *n* units more valuable than *Y*.

This is just like the previous principle, except that it specifies *how much* more valuable on option is than the other. It says, for example that SP* is .4 units more valuable than SP.

It may seem confused to hold that one infinitely valuable option is *n* units more valuable than another infinitely valuable option, but it is not. For *finitely valuable* options, *X* and *Y*, if *X* is

[8] For a closely related principle in value theory (but without probabilities), see Lauwers and Vallentyne (2004).

n units more valuable than *Y*, then the value of *X* is *n* units higher than the value of *Y*. This relationship, however, does not hold for infinitely valuable options. To say that an option is infinitely valuable is not to assign it a specific value. It is merely to say that it is more valuable than any finitely valuable option. Infinitely valuable options need not be equally valuable (and usually are not). Often they are incomparable, but sometimes one is more valuable another (e.g., when one dominates the other). Moreover, it is sometimes possible to say how much more valuable one infinitely valuable option is compared with another. To say that *X* is *n* units more valuable than *Y* is just to say that *X* is equally valuable with the option obtained by increasing, for each state, *Y*'s payoffs by *n* units. For example, above, SP* is .4 units more valuable than SP, in the sense that it would be worth paying .4 units to exchange SP for SP*. There is no claim that adding .4 to infinity is somehow greater than infinity. Once that is understood, there is no incoherence.

To see the need for the second strengthening, consider:

Relative Weak Expectations

Probability:	1/2	1/4	1/8	1/16 …	$1/2^{2n-1}$	$1/2^{2n}$ …
State:	s_1	s_2	s_3	s_4 …	s_{2n-1}	s_{2n} …
SP	2	4	8	16 …	2^{2n-1}	2^{2n} …
Q	2+2	4–4/2	8+8/3	16–16/4 …	$2^{2n-1}+(2^{2n-1})/(2n-1)$	$2^{2n}-(2^{2n})/2n$ …
Q–SP	2	–4/2	8/3	–16/4 …	$(2^{2n-1})/(2n-1)$	$-(2^{2n})/2n$ …

In this example, for Q–SP (which is just Pasadena), the sum of the probability-weighted payoffs is infinite for both the positive payoffs and for the negative payoffs. Thus, Q–SP has no standard expected value, and Cardinal Relative Expectations is silent. Note, however, that Q–SP has a *weak expected value* of log(2). We believe that that is sufficient to evaluate Q as being log(2) units more valuable than SP.

More generally, we believe that the following principle is plausible:

Cardinal Relative Weak Expectations. If the *weak expected value* of $X–Y$ is n, then X is n units more valuable than Y.

This principle replaces the appeal to the standard expected value of $X–Y$ with an appeal to the weak expected value of $X–Y$. It rightly assesses Q as log(2) units more valuable than SP. It will often give more determinate assessments of the comparative value of two options than does Cardinal Interval Weak Expectations (e.g., that X is n units more valuable, rather than being not less than n-1 units more valuable and not more than n+1 units more valuable). This is not a conflict. It is simply a case of being more determinate.

In the above example, the two options each have infinite weak expected value. A similar example could be given where neither option has a weak expected value but their difference does (e.g., the same difference as Q–SP above).

To see the need for one final strengthening of the relative expectations principle, consider the following example:

Oscillating Differences

Probability:	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	… $1/2^n$ …
State:	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	… s_n …

SP	2	4	8	16	32	64	128	256	… 2^n …
U	$2+a_1$	$4+a_2$	$8+a_3$	$16+a_4$	$32+a_5$	$64+a_6$	$128+a_7$	$256+a_8$	… 2^n+a_n …
U–SP	2	$-2^2/2$	$-2^3/3$	$2^4/4$	$2^5/5$	$2^6/6$	$2^7/7$	$-2^8/8$	… a_n …

Here (1) SP is St. Petersburg, (2) U is defined to have the same payoffs as SP but increased by a_n for state s_n, and (3) the a_i are the same as the Pasadena payoffs for state s_i, except that the signs are adjusted to ensure that E([U–SP]$_n$) is inclusively between 0 and 1. This ensures that E([U–SP]$_n$) oscillates between 0 and 1, as *n* goes to infinity. (The set up here is similar to that of Oscillating Weak Expected Value in the section on indeterminate value.)

Here, SP and U each have infinite weak expected value, but U–SP has no weak expected value, given that E([U–SP]$_n$) does not have a limit, as *n* goes to infinity. Thus, Cardinal Relative Weak Expectations is silent. Nonetheless, it is plausible, we claim, that U is 0 to 1 units more valuable than SP. Indeed, note that, although U–SP has no weak expected value, it does have a weak expected interval value of [0, 1], where (as defined above) this means that 0 is the greatest lower bound on the value of *k* for which lim pr(Ave(X,n)≥k) = 1, and (2) 1 is the smallest upper bound on the value of *k* for which lim pr(Ave(X,n)≤k) = 1.

Consider, then, the following principle, where to say that *X* is *m to n units more valuable* than *Y* is to say that (1) *m* is the greatest lower bound on the value by which the value of *X* exceeds that of *Y*, and (2) *n* is the least upper bound on the value by which the value of *X* exceeds that of *Y*:

Cardinal Relative Interval Weak Expectations. If *X*–*Y* has a weak expected interval value of [*m*, *n*], then *X* is *m* to *n* units more valuable than *Y*.

As indicated above, when m is negative, to say that X is m units more valuable than Y is to say that X is at least as valuable as the option obtained by decreasing payoffs by |m| units.

Cardinal Relative Interval Weak Expectations is equivalent to Cardinal Relative Weak Expectations in the special case where $X–Y$ has a weak expected interval value [k, k], for k finite or infinite. Moreover, this principle entails the conjunction of Cardinal Interval Weak Expectations with Cardinal Relative Weak Expectations. It entails, that is, that, if X and Y each have a weak expected interval values of $[x_1,x_2]$ and $[y_1,y_2]$, then X is not less than x_1-y_2 units more valuable than Y, and not more than x_2-y_1 units more valuable (proof in footnote).[9]

Cardinal Relative Interval Weak Expectations determines when one option is m to n units more valuable than another. It does not, however, assess the *non-relative* value of any option. It does not entail that Pasadena, for example, has value log(2). It thus does not entail Weak

---

[9] Proof: Suppose that X and Y have interval values $[x_1,x_2]$ and $[y_1,y_2]$ respectively. It follows that the probability that $\mathrm{Ave}(X,n)\geq x_1$ converges to one as n goes to infinity and that the probability that $\mathrm{Ave}(–Y,n)>–y_2$ converges to one as n goes to infinity. Consider a sufficiently large n such that both events have a probability of at least $1–e/2$, for some positive e less than 1. It follows from Bonferroni's inequality that both events occur simultaneously with probability $1–e$. Hence, for this large n, the probability that [$\mathrm{Ave}(X,n)>x_1$ and $\mathrm{Ave}(–Y,n)>–y_2$] is greater than $1–e$. It follows that the probability that $\mathrm{Ave}(X–Y,n)>x_1–y_2$ is greater than $1–e$. Given that this argument holds for every strictly positive e less than one, it follows that the lower bound of the weak expected value of $X–Y$ is not less than $x_1–y_2$. In a similar way, one can prove that the upper bound is not more than $x_2–y_1$. QED.

Expectations (which assigns each option its weak expected value, if it has one). If, however, we add the following uncontroversial principle, then Weak Expectations will follow:

Zero Value for Zero Option. An option that has a payoff of zero for all basic states has value 0.

Given this principle, Cardinal Relative Interval Weak Expectations entails Weak Expectations, since if *X* has a weak expected value of *n*, then *X* is *n* units more valuable than the zero option. Given that the latter has value 0, the former has value *n*, as required.

Cardinal Relative Interval Weak Expectations is, we think, highly plausible. We shall now consider two extensions and argue that they should be rejected.

7. Rejecting Basic-State Isomorphism and Stochastic Dominance

We shall here briefly consider and reject (1) a principle holding that isomorphic options are equally valuable, and (2) a principle holding that, if one option stochastically dominates another, then it is more valuable.

7.1 Rejecting Basic-State Isomorphism

To see that an additional principle may be needed, consider the following version of an example provided in correspondence by James Joyce, as noted in Colyvan and Hájek (2015). There are two St. Petersburg options, SP1 and SP2. Their payoffs are both based on the number of flips it takes for given coin to first come up heads. Let this be *n*. The only difference is that a second coin is also flipped, and if H occurs SP1 gets the payoff of 2^{n+1} and SP2 gets 2, and if T occurs the payoffs are switched. There is also a third option, SP1+, which has the same payoff as SP1, if the second coin comes up T, and has a payoff one greater than SP1, if the second coin comes up H. This can be represented as follows, where (n,H) and (n,T) are the states where the first heads occurs on the n-th flip of the first coin and the second coin comes up H, or respectively, T.

Three St. Petersburg-like Options

Probability:	1/4	1/4	1/8	1/8	1/16	1/16	…	$1/2^{n+1}$	$1/2^{n+1}$	…
State:	(1,H)	(1,T)	(2,H)	(2,T)	(3,H)	(3,T)	…	(n,H)	(n,T)	…
SP1	4	2	8	2	16	2	…	2^{n+1}	2	…
SP2	2	4	2	8	2	16	…	2	2^{n+1}	…
$SP1^{+}$	4+1	2	8+1	2	16+1	2	…	$2^{n+1}+1$	2	…
SP1–SP2	2	–2	6	–6	14	–14	…	$2^{n+1}–2$	$–2^{n+1}+2$	…
$SP1^{+}$–SP1	1	0	1	0	1	0	…	1	0	…
$SP1^{+}$–SP2	3	–2	7	–6	15	–14	…	$2^{n+1}–1$	$2–(2^{n+1})$	…

$SP1^{+}$–SP1 has a standard (and hence weak) expected value of 1/2 (= 1/4 + 1/8 + 1/16 ...). Hence, $SP1^{+}$ is 1/2 a unit more valuable than SP1. By contrast, $SP1^{+}$–SP2 has interval weak expectation [-∞,∞], and thus appeal to relative weak expectations provides no basis for assessing SP1+ compared with SP2. In response to this example, Colyvan and Hájek (2015: 15) write: “So, it looks like either the states have to be individuated by something more than their probabilities, or we have to swallow the above paradoxical results about the RET’s [Relative Expectation Theory’s] verdicts about the comparisons between A [=$SP1^{+}$], B [=SP2] and B* [=SP1] [i.e., that $SP1^{+}$ is more valuable than SP1 but not more valuable than SP2].”

A natural thought is to say that SP1 and SP2 are equally valuable because they have the same payoff structure. The payoff for X (respectively: Y) under (n,T) is the same as that for Y (respectively: X) under the equally probable (n,H). Thus, if SP1+ is 1/2 a unit more valuable than SP1, then it is also 1/2 a unit more valuable than SP2.

More exactly, SP1 and SP2 may seem equally valuable, given that they are *basic-state isomorphic* in the following sense: there is a way of pairing up the SP1's basic states with SP2's basic states (i.e., a 1-to-1 mapping from the former onto the latter), such that (1) the probability of each basic state is the same as its partner's probability, and (2) SP1's payoff under a basic state is the same as SP2's payoff under that basic state's partner. In the above example, SP1 and SP2 are basic-state isomorphic. The only difference between the two is that SP1 has payoffs of 2^{n+1} for the H-states and a payoff of 2 for the T-states whereas the opposite is true for SP2. For example, the payoff for SP1 under (1,H) is the same as the payoff for SP2 under (1,T), and (1,H) and (1,T) have the same probability, and so on.

Given that SP1 and SP2 are basic-state isomorphic, the following principle entails that they are equally valuable:

Basic-State Isomorphism. If two options are basic-state isomorphic, then they are equally valuable.

This is a plausible, but redundant, principle when both options have *finite* weak expected value. It is not, however, a redundant principle when both options lack finite weak expected value. In the example above, SP1–SP2 has interval weak expected value $[-\infty,\infty]$, and thus Cardinal Relative Interval Weak Expectations is silent. Nonetheless, it may seem plausible that

SP1 and SP2 are equally valuable. Basic-State Isomorphism captures that intuition.[10]

Basic-State Isomorphism is, however, mistaken. A brilliant example from Seidenfeld et al. (2009) shows this. Note that the payoff for SP1 above does not depend on *n*, if the second coin-flip comes up T. Thus, it can be represented as:

Probability:	1/2	1/4	1/8	1/16	…	$1/2^{n+1}$	…
State:	T	(1,H)	(2,H)	(3,H)	…	(n,H)	…
SP1	2	4	8	16	…	2^{n+1}	…

Likewise, the payoff for SP2 above does not depend on *n*, if the second coin-flip comes up H. Thus, it can be represented as:

Probability:	1/2	1/4	1/8	1/16	…	$1/2^{n+1}$	…
State:	H	(1,T)	(2,T)	(3,T)	…	(n,T)	…
SP2	2	4	8	16	…	2^{n+1}	…

If the states listed for SP1 and SP2 are their basic states, then SP1 and SP2 are basic-state isomorphic and thus judged equally valuable by the above principle. This may not problematic, but the next implication is.

[10] Easwaran (2014b) develops a very elegant theory of expected utility by appealing to something like Basic-State Isomorphism (or what he calls “measure-preserving correspondences” between state spaces).

Consider now a third St. Petersburg option, W, defined by the same coin flips, where if the first heads of the first coin occurs on the *n*-th flip, its payoff is 2^n. Unlike, SP1 and SP2, the outcome of the flip of the second coin is irrelevant to its payoffs. This can be represented as follows:

Probability:	1/2	1/4	1/8	1/16	…	$1/2^n$	…
State:	(1)	(2)	(3)	(4)	…	(n)	…
W	2	4	8	16	…	2^n	…

Again, if these are W's basic states, then W is basic-state isomorphic with SP1 and with SP2. W's (1) has the same probability as SP1's (H), etc. The isomorphism principle then entails that W is equally valuable with SP1 and with SP2. This, however, is problematic, as becomes obvious when they are represented in the same state-space (recall that all three depend on the flips of the same two coins):

Three More St. Petersburg Options

Probability:	1/4	1/4	1/8	1/8	1/16	1/16	…	$1/2^{n+1}$	$1/2^{n+1}$	…
State:	(1,H)	(1,T)	(2,H)	(2,T)	(3,H)	(3,T)	…	(n,H)	(n,T)	…
W	2	2	4	4	8	8	...	2^n	2^n	…
SP1	4	2	8	2	16	2	…	2^{n+1}	2	…
SP2	2	4	2	8	2	16	…	2	2^{n+1}	…
(SP1+SP2)/2	3	3	5	5	9	9	...	2^n+1	2^n+1	...

Here, we can see that (SP1+SP2)/2 state-dominates W in the strong sense that its payoff is greater than that of W *in every single basic state* (in their shared state space). This is not possible if W is equally valuable with both SP1 and SP2. So, the above isomorphism principle must be rejected.[11]

Above W appears not to be basic-state isomorphic with SP1 (and SP2). It has a payoff of 2 in only two of the listed states, whereas SP1 has a payoff 2 in infinitely many listed states. This, however, is deceptive. The above display shows irrelevant distinctions. For example, for W, the distinction between the H and T states is irrelevant. Its payoffs are fully determined by the value of n. Likewise, for SP1, for the T states, the distinction among the different *n* states is irrelevant. Its payoff is always 2 for a T state. Once, these irrelevant distinctions are removed, the payoff structure is the same.

We suspect that some kind of isomorphism principle is valid, and that it will judge SP1 and SP2 (but not W) equally valuable. We believe, however, that it will involve a deeper account of what an option's basic states are (e.g., by appeal to its underlying tree structure) and a more restricted version of isomorphism (e.g., applying only when the tree/state space is the same for both options). That, however, is a project for another paper. The above is sufficient to show that

[11] We are thus agreeing with Seidenfeld et al. (2009) and Smith (2014) in rejecting what Smith (465) calls "the principle of compositionality": "the value which a rational agent places [or at least should place] on a gamble is [solely] a function of the values which she places on the possible outcomes of the gamble, together with the probabilities assigned to those outcomes by the gamble." W and SP1 assign the same probabilities to the same payoffs but are not equally valuable.

the above unsophisticated isomorphism principle must be rejected.

7.2 Rejecting Stochastic Dominance

Let us now consider a principle of stochastic dominance (defined below). To see the apparent need for such a principle, consider the following simplified version of an example from Peterson (2011), where SP1 is defined as above, and SP3 is defined below.

Stochastic Dominance

Probability:	1/4	1/4	1/8	1/8	1/16	1/16	…	$1/2^{n+1}$	$1/2^{n+1}$	…
State:	(1,H)	(1,T)	(2,H)	(2,T)	(3,H)	(3,T)	…	(n,H)	(n,T)	…
SP1	4	2	8	2	16	2	…	2^{n+1}	2	…
SP3	0	2	0	4	0	8	…	0	2^n	…

Here SP1 *stochastically dominates* SP3 in the sense that its distribution can be obtained from that of SP3 by shifting probability from lower payoffs to higher payoffs (e.g., the 1/2 probability of 0 in SP3 is shifted upwards as follows: 1/4 to 2, 1/8 to 4, ….). It seems plausible that a stochastically dominating option is more valuable than the option it dominates. This is always true when both options have finite weak expectations. Colyvan and Hájek (2015: 14) note that relative (standard) expectation theory is strangely silent about cases of stochastic dominance. The stronger Cardinal Relative Interval Weak Expectations is also silent, since SP1–SP3 has value $[-\infty,\infty]$.

It is therefore tempting to endorse:

Stochastic Dominance. If one option stochastically dominates another, then it is more valuable than the other.

This principle, however, must be rejected. To see this consider, the following example, where SP1 and SP2 are as above, and W+ is the same as W above, except that its payoff for (1,H) and (1,T) is increased by .1:

Stochastic Dominance Rejected

Probability:	1/4	1/4	1/8	1/8	1/16	1/16	…	$1/2^{n+1}$	$1/2^{n+1}$	…
State:	(1,H)	(1,T)	(2,H)	(2,T)	(3,H)	(3,T)	…	(n,H)	(n,T)	…
W+	2.1	2.1	4	4	8	8	…	2^n	2^n	...
SP1	4	2	8	2	16	2	…	2^{n+1}	2	…
SP2	2	4	2	8	2	16	…	2	2^n	…
(SP1+SP2)/2	3	3	5	5	9	9	...	2^{n+1}	2^{n+1}	...

W+ stochastically dominates SP1 and SP2 (since it shifts the .5 probability from 2 to 2.1). Stochastic Dominance then entails that W+ is more valuable than SP1 and more valuable than SP2. This, however, is problematic, given that (SP1+SP2)/2 has a higher payoff than W+ in every single state. If W+ is more valuable than SP1 and more valuable than SP2, then it can't be case that (SP1+SP2)/2 is more valuable than W+. Thus, Stochastic Dominance must be rejected.

Stochastic dominance is based solely on the probabilities of payoffs and thus lacks relevant (state space) information. This does not cause a problem where both options have finite weak expectations. It does, however, have unacceptable implications when neither has finite weak expectations.

8. Conclusion

We have limited our attention throughout to state spaces with *countably-many* basic states and *countably additive* probability functions (i.e., the probability of a union of a countable number of basic states is the sum of their individual probabilities). We have further limited our attention to cases where probabilities and payoffs are fully defined finite standard numbers. Things are significantly more complex when these restrictions are dropped.

We have developed, with motivation but not compelling argument, several principles for the evaluation of options that have no finite standard expected value. The case arises when the sum of the probability-weighted values of an option is either ill defined (because the sum depends on the order in which the terms are added together) or infinite. Standard decision theory is silent about the evaluation of such options. Our project has been to extend the domain of evaluation.

First, we endorsed the proposal of Easwaran (2008) to evaluate options on the basis of their finite *weak* expected value, if they have one. We then extended that to include *infinite* weak expected value.

Second, we extended the relevant notion of weak expected value to *interval value,* where this is defined as the closed interval $[x_1,x_2]$ such that: (1) x_1 is the greatest lower bound on the value of k for which lim pr(Ave(X,n)$\geq k$) = 1, and (2) x_2 is the smallest upper bound on the value of k for which lim pr(Ave(X,n)$\leq k$) = 1. Many options have an interval value (e.g., [1,3]) even though they have no weak expected value. An option with an interval value of [4,6] is more valuable than an option with interval value [1,3]. Indeed it is 1 to 5 units more valuable.

Third, for options defined on the same state space, we endorsed the proposal of Colyvan (2008) to evaluate an option, *X*, as at least as valuable as an option *Y*, if the standard expected value of *X*–*Y* exists and is non-negative (where *X*–*Y* is defined over basic states) . We then extended that principle (1) to be a cardinal principle that states how many more units more valuable *X* is, (2) to base the evaluation on weak (rather than standard) expected value, and (3) to apply in certain cases where *X*–*Y* has no weak expected value but has an *interval* value. When complemented with the uncontroversial Zero Value for Zero Option, this entails the Cardinal Interval Weak Expectations (that options with interval weak expectations have that value).

We also considered two seemingly plausible principles that make no appeal to weak expectations: Basic-State Isomorphism and Stochastic Dominance. Both were found to have unacceptable implications. We suspect that a weaker version of Basic-State Isomorphism may be plausible, but we have left that exploration for another occasion.

In closing, we shall briefly mention three other principles that we have not discussed. Gwiazda (2014) has proposed evaluating all options on the basis of the limit, as *n* goes to positive infinity, of the standard expected value of (X_n), where X_n truncates *X* to zero all payoffs with absolute value greater than n. This is clearly correct, where *X* has thin tails, and thus a weak expected value, but, as Easwaran (2014a) notes, it seems suspect in most other cases. Easwaran proposes, more cautiously, appealing to this value in the case where the limit does not change when a constant is subtracted (or added) to all payoffs (i.e., when changing zero-point for value does not change the limit). As Easwaran notes, however, one might question why the summation of the probability weighted values is done *symmetrically from the zero point* for payoffs. To this, we add that one might also question why the summation is done on the basis *payoff intervals* rather than arbitrary sets of payoff values. Finally, Bartha (2015) has proposed Relative Utility

Theory, which appeals to *ratios* of payoffs rather than *differences* of payoffs (as the relative expectation principles do). With respect to the domain of application of this paper (finite standard probabilities and payoffs), we believe that Bartha's approach does not make any comparative assessments beyond those we endorse above. Moreover, Relative Utility Theory has the disadvantage of relativization to an arbitrary third option.[12] Each of these proposal merits careful examination, but we cannot give it here.

Our principles are, of course, controversial. We hope, however, that we have motivated them enough to be taken seriously.

ACKNOWLEDGEMENTS: For very helpful comments on this, or (significantly different) earlier versions of this paper, we thank: Alan Hajek, Paul Weirich, Kenny Easwaran, Toby Ord, Teddy Seidenfeld, Paul Bartham, and three anonymous referees for this journal.

[12] For elaboration of these issues, see Colyvan and Hájek (2015).

REFERENCES

Bartha, P. 2015. Making do without expectations. *Mind* forthcoming.

Baum, L.E. 1963. On convergence to + ∞ in the law of large numbers. *The Annals of Mathematical Statistics* 34: 219--222.

Colyvan, M. 2008. Relative expectation theory. *Journal of Philosophy* 105: 37--44.

Colyvan, M. and A. Hájek. 2015. 'Making ado without expectations. *Mind* forthcoming.

Derman C., and H. Robbins. 1955. The strong law of large numbers when the first moment does not exist. *Proceedings of the National Academy of Sciences of the United States of America* 41: 586--587.

Durrett, R. 2005. *Probability: Theory and Examples,* 3rd edition. Belmont, CA: Thomson Brooks/Cole.

Easwaran, K. 2008. Strong and weak expectations. *Mind* 117: 633--641.

Easwaran, K. 2014a. Principal values and weak expectations. *Mind* 123: 517--531.

Easwaran, K. 2014b. Decision theory without representation theorems. *Philosophers' Imprint* 14: 1--30.

Erickson, K.B. 1973. The strong law of large numbers when the mean is undefined. *Transactions of the American Mathematical Society* 185: 371--381.

Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*, volume 1. New York: John Wiley & Sons.

Feller, W. 1971. *An Introduction to Probability Theory and Its Applications*, volume 2. New York: John Wiley & Sons.

Fine, T.L. 2008. Evaluating the Pasadena, Altadena, and St. Petersburg gambles. *Mind* 117: 613--32.

Fishburn, P. 1985. *Interval Orders and Interval Graphs*. New York: John Wiley & Sons.

Gawiazda, J. 2014. Orderly expectations. *Mind* 123: 503--516.

Hájek, A. 2009. All values great and small. Manuscript.

Hájek, A. and H, Nover. 2006. Perplexing expectations. *Mind* 115: 703--720.

Hájek, A. and H. Nover. 2008. Complex expectations. *Mind* 117: 643--664.

Hájek, A. and M. Smithson. 2012. Rationality and indeterminate probabilities. *Synthese* 187: 33--48.

Lauwers, L. and P. Vallentyne. 2004. Infinite utilitarianism: more is always better. *Economics and Philosophy* 20: 307--330.

Lauwers, L. and P. Vallentyne. 2015. A tree can make a difference. In progress.

Nover, H. and A. Hájek. 2004. Vexing expectations. *Mind* 113: 237--249.

Peterson, M. 2011. A new twist to the St. Petersburg paradox. *Journal of Philosophy* 108: 697--99.

Seidenfeld, T., M.J. Schervish, and J.B. Kadane. 2009. Preference for equivalent random variables: a price for unbounded utilities. *Journal of Mathematical Economics* 45: 329--340.

Smith, N. 2014. Is evaluative compositionality a requirement of rationality? *Mind* 123: 457--502.

Sprenger, J. and R. Heesen. 2011. The bounded strength of weak expectations. *Mind* 120: 819--832.

BIOGRAPHICAL INFORMATION

Luc Lauwers is professor at the Katholieke Universiteit Leuven, Center of Economic Studies. He works on social choice, cooperative game theory, and decision theory. He is member of the

editorial board of *Mathematical Social Sciences*.

Peter Vallentyne is Florence G. Kline Professor of Philosophy at the University of Missouri. He writes on issues of liberty and equality in the theory of justice (and left-libertarianism in particular), enforcement rights (rights to protect primary rights), and decision theory where there is no finite standard value. He is an associate editor of *Ethics*.